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Solving Families of Simultaneous Pell Equations

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If a and b are distinct positive integers then a previous result of the author implies that the simultaneous Diophantine equations

$$x^2 - az^2 = y^2 - bz^2 = 1$$

possess at most 3 solutions in positive integers (x, y, z). On the other hand, there are infinite families of distinct integers (a, b) for which the above equations have at least 2 positive solutions. For each such family, we prove that there are precisely 2 solutions, with the possible exceptions of finitely many pairs (a, b). Since these families provide essentially the only pairs (a, b) for which the above equations are known to have more than a single solution (in positive (x, y, z)), this lends support to the conjecture that the number of such solutions to the above equations is $\leqslant 2$ in all cases. © 1997 Academic Press

1. INTRODUCTION

In this note, we consider the simultaneous Diophantine equations

$$x^2 - az^2 = y^2 - bz^2 = 1, \tag{1.1}$$

where a and b are distinct nonzero integers. These and related equations arise in connection with a variety of classical problems on polygonal numbers (see e.g. [3]), from consideration of elliptic curves with good reduction away from 2 (see [8]) and in the construction of P_t sets (see [4] for relevant definitions and a more complete bibliography of the abundant literature on the subject). As has been noted by Ono [7], positive solutions (x, y, z) to (1.1) imply the existence of rational points of infinite order on the elliptic curve

$$Y^2 = X(X + a)(X + b),$$

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a fact used in [7] to describe certain families of (a, b) for which (1.1) possesses no nontrivial solutions.

Let us denote by $N(a, b)$ the number of solutions to (1.1) in positive integers (x, y, z). In [2], the author, sharpening work of Masser and Rickert [5], proved

THEOREM 1.1. *If a and b are distinct nonzero integers, then $N(a, b) \leqslant 3$.*

In the direction of lower bounds, let l and m be integers with $l, m \geqslant 2$ and set

$$n(l, m) = \frac{\alpha^{2l} - \alpha^{-2l}}{4\sqrt{m^2-1}}$$

with $\alpha = m + \sqrt{m^2-1}$. It follows that $N_{l,m} = N(m^2-1,\ n(l, m)^2 - 1) \geqslant 2$, corresponding to the solutions

$$(x_1, y_1, z_1) = (m, n(l, m), 1)$$

and

$$(x_2, y_2, z_2) = \left(2n(l, m)m - \frac{n(l, m)}{m} - \frac{n(l-1, m)}{m}, 2n(l, m)^2 - 1, 2n(l, m)\right)$$

to (1.1) (it is readily seen that m divides $n(l, m)$ for all l whence the second solution is in fact integral). Thus Theorem 1.1 is not too far from the truth.

The purpose of the present paper is to provide some evidence for the following

Conjecture 1.2. If a and b are distinct nonzero integers, then $N(a, b) \leqslant 2$.

Since the only pairs (a, b) known for which (1.1) possesses at least two positive solutions are equivalent to $(m^2-1,\ n(l, m)^2-1)$ for certain l and m (in a sense we will make precise in the next section), a useful first step in proving this conjecture would be to show that $N_{l,m} = 2$ for all $l, m \geqslant 2$. Towards this end, we find

THEOREM 1.3. *If l and m are integers with $l \geqslant 2$ and $m \geqslant 2 \times 10^7 \sqrt{l} \log^2 l$, then $N_{l,m} = 2$.*

In essence, this result provides an (almost complete) solution to infinite families of simultaneous Pell equations, somewhat analogous to the solution of families of Thue equations due to Thomas [9] (see also [6] and [10]).

2. SOME TOOLS

Suppose that $b > a \geqslant 2$ are nonsquare integers and let α and β denote the fundamental solutions to $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$ respectively (i.e. let $\alpha = a_1 + a_2 \sqrt{a}$ and $\beta = b_1 + b_2 \sqrt{b}$ where $(x, z) = (a_1, a_2)$ and $(y, z) = (b_1, b_2)$ are the smallest positive integer solutions to $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$, respectively). It follows that if (x_i, y_i, z_i) is a positive solution to (1.1), then

$$z_i = \frac{\alpha^{j_i} - \alpha^{-j_i}}{2\sqrt{a}} = \frac{\beta^{k_i} - \beta^{-k_i}}{2\sqrt{b}} \tag{2.1}$$

for positive integers j_i and k_i. In [2], we established the following gap principle:

Lemma 2.1. *If* (x_i, y_i, z_i) *are distinct positive solutions to* (1.1) *for* $1 \leqslant i \leqslant 3$, *with* j_i *and* k_i *as defined above, then*

$$(k_3 - k_2)(k_2 - k_1) > \alpha^{2j_1}.$$

The families of (a, b) described in the previous section with $N(a, b) \geqslant 2$ correspond to prescribing $j_1 = k_1 = 1$ and $j_2 = 2l$, $k_2 = 2$. Let us further note, at this juncture, that the restriction to a and b of the form $a = m^2 - 1$ and $b = n^2 - 1$ is without loss of generality (provided $N(a, b) \geqslant 1$). To see this, suppose that $\alpha = a_1 + a_2 \sqrt{a}$ and $B = b_1 + b_2 \sqrt{b}$ are fundamental solutions to $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$ respectively. Then solutions to $x^2 - az^2 = 1$ correspond to those to $x^2 - (a_1^2 - 1) z_1^2 = 1$ via $z = a_2 z_1$. If ba_2^2 is of the form $n^2 - 1$ for some n, we reach the desired conclusion. If not, then if $\gamma = c_1 + c_2 \sqrt{ba_2^2}$ is the fundamental solution to $x^2 - (ba_2^2) z_1^2 = 1$, we have tha $z_1 = c_2 z_2$ for integral z_2, etc. Iterating this argument by considering, in succession, the equations derived from $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$, at some stage, since we suppose that the equations (1.1) have a positive solution, this procedure must terminate. This establishes the claim.

For the remainder of the paper, we will restrict our attention to the aforementioned families (where we have $N_{l,m} \geqslant 2$). Let us suppose that there exists a third positive solution (x_3, y_3, z_3) to (1.1), with corresponding j_3 and k_3. Lemma 2.1 therefore implies that

$$k_3 - 2 > (m + \sqrt{m^2 - 1})^2,$$

whence we may readily show that

$$k_3 \geqslant 4m^2 > \alpha^2, \tag{2.2}$$

since k_3 is integral. To use this inequality, we require an estimate for linear forms in the logarithms of (three) algebraic numbers, say the following recent result due to Voutier [11]:

THEOREM 2.2. *Let l_1, l_2 and l_3 be logarithms of three non-zero algebraic numbers α_1, α_2 and α_3 and put*

$$D = [\mathbf{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbf{Q}]/[\mathbf{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbf{R}].$$

Let b_1, b_2 and b_3 be nonzero integers with $(b_1, b_2, b_3) = 1$ and let A_1, A_2, A_3, B, $E > 1$ and $k \geqslant 1$ be positive real numbers satisfying

$$\log A_i \geqslant \max\left\{\frac{\log E}{D}, h(\alpha_i), \frac{E\,|l_i|}{D}\right\} \qquad \text{for} \quad 1 \leqslant i \leqslant 3$$

and

$$B \geqslant \max\left\{2, E^{k/D}, \max_{1 \leqslant i \neq j \leqslant 3} \frac{|b_i| \log E}{D \log A_j} + \frac{|b_j| \log E}{D \log A_i}\right\}.$$

If

$$\Lambda = b_1 l_1 + b_2 l_2 + b_3 l_3$$

and the l_i's are linearly-independent over $\mathbf{Q}$, *then*

$$|\Lambda| > \exp(-C(k, E)\, D^5 (\log B)^2 \log A_1 \log A_2 \log A_3),$$

where

$$C(k, E) = \frac{1447000}{(\log E)^4}\left(1 + \frac{2.6}{k} + \frac{1.7}{k^2}\right)\left(1 + \frac{1}{3E}\right)^3.$$

Here, $h(\alpha)$ denotes the standard logarithmic Weil height of an algebraic number α. A result of this sort allows one to effectively solve any given system of equations of the form (1.1), in conjunction with techniques from computational Diophantine approximation (see e.g. [1] where it is shown that (1.1) has at most one positive solution for $2 \leqslant a < b \leqslant 200$). Together with (2.2), it will enable us to tackle whole families of pairs (a, b).

3. PROOF OF THEOREM 1.3

We apply Theorem 2.2 with

$$\alpha_1 = \sqrt{b/a},\ \alpha_2 = \alpha,\ \alpha_3 = \beta,\ b_1 = 1,\ b_2 = j_3, \text{ and } b_3 = -k_3.$$

Setting $E=e$ (for simplicity; we can obtain somewhat sharper results with a more carefully chosen E), we will suppose that $D=4$ (since we clearly have $D \leqslant 4$ and, if the case $D=2$, the bounds we obtain are in fact stronger in all situations). It follows that we may take

$$\log A_1 = \frac{1}{2} \log b, \qquad \log A_2 = \frac{e}{4} \log \alpha, \qquad \log A_3 = \frac{e}{4} \log \beta,$$

and

$$B \geqslant \max\left\{ e^{k/4}, \frac{j_3}{\log A_3} + \frac{k_3}{\log A_2} \right\}.$$

It is readily observed that $j_3/\log A_3 < k_3/\log A_2$ and since (2.2) and the hypotheses of Theorem 1.3 imply that $k_3 > 7 \times 10^{14}$, we may choose $B = k_3$ and $k = 136$. Theorem 2.2 therefore implies that

$$\log |\Lambda| > -4.94 \times 10^8 \log^2(k_3) \log(b) \log(\alpha) \log(\beta).$$

On the other hand, arguing as in [2], (2.1) yields

$$\log |\Lambda| < -2k_3 \log \beta + \log(b/a) < -1.99 \times k_3 \log \beta, \tag{3.1}$$

where the last inequality follows from (2.2) and the lower bound for k_3. We therefore have

$$\frac{k_3}{\log^2 k_3} < 2.49 \times 10^8 \log(b) \log(\alpha).$$

Since one has $b < 4l \log \alpha$, (2.2) implies that

$$\frac{m^2}{\log^4(2m)} < 10^9 l,$$

contradicting the fact that

$$m \geqslant 2 \times 10^7 \sqrt{l} \log^2 l.$$

This completes the proof of Theorem 1.3.

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